Lecture-8

Boundary condition.
Electrostatic boundary value problems

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4. Boundary Conditions for Electrostatic Fields

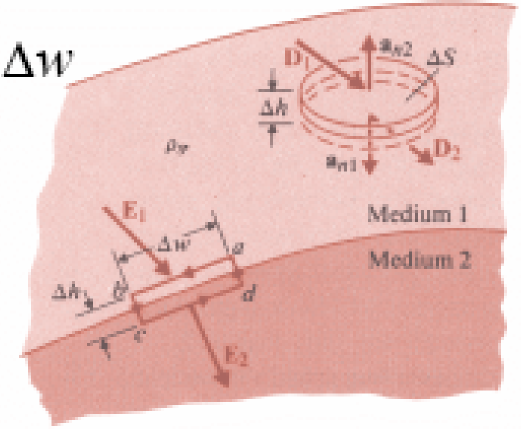
Electromagnetic problems often involve media with different physical properties and require the knowledge of the relations of the field quantities at an interface between two media. For instance, we may wish to determine how the **E** and **D** vectors change in crossing an interface. We already know the boundary conditions that must be satisfied at a conductor/free space interface. These conditions have been given in previous section. We now consider an interface between two general media shown in the figure.

Let us construct a small path *abcda* with sides *ab* and *cd* in media 1 and 2, respectively, both being parallel to the interface and equal to Δw. We apply

$$\oint_C \bar{E} \cdot d\bar{l} = 0$$

to the path. If we let sides *bc*=*da*= Δh approach zero, we have

$$\oint_{abcda} \bar{E} \cdot d\bar{l} = \bar{E}_1 \cdot \Delta w + \bar{E}_2 \cdot (-\Delta w) = E_{1t} \cdot \Delta w - E_{2t} \cdot \Delta w = 0$$



Therefore,

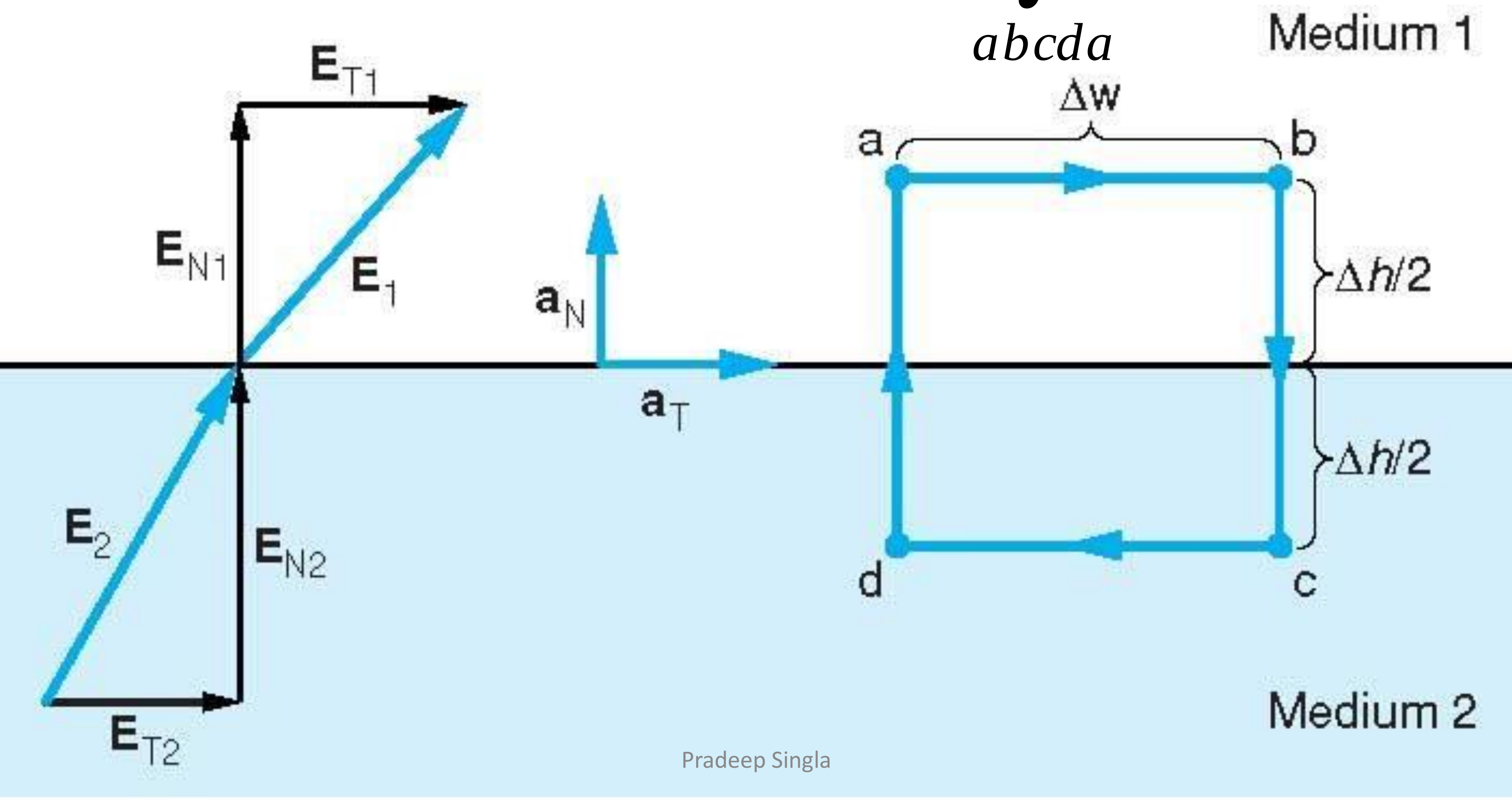
$$E_{1t} = E_{2t} \quad (V/m)$$

which states that the tangential component of an ***E*** field is continuous across an interface.

Tangential Components of the Electric Field

CONDITION I (tangential components)

$$\oint_{abcda} \overline{E} \bullet d\bar{l} = 0$$



$$\int_{a}^{b} \overline{E} \bullet d\bar{l} + \int_{b}^{c} \overline{E} \bullet d\bar{l} + \int_{c}^{d} \overline{E} \bullet d\bar{l} + \int_{d}^{a} \overline{E} \bullet d\bar{l} = 0$$

$$\int_{0}^{\Delta w} \overline{E} \bullet d\bar{l} + \int_{\Delta h/2}^{0} \overline{E} \bullet d\bar{l} + \int_{0}^{-\Delta h/2} \overline{E} \bullet d\bar{l} + \int_{\Delta w}^{0} \overline{E} \bullet d\bar{l} +$$

$$+ \int_{-\Delta h/2}^{0} \overline{E} \bullet d\bar{l} + \int_{0}^{\Delta h/2} \overline{E} \bullet d\bar{l} = 0$$

$$\int_{0}^{\Delta w} E_{T_1}\,\overline{a_T}\,dl\,\overline{a_T} + \int_{\Delta h/2}^{0} E_{N_1}\,\overline{a_n}\,dl\,\overline{a_n} + \int_{0}^{-\Delta h/2} E_{N_2}\,\overline{a_n}\,dl\,\overline{a_n}$$

$$+ \int_{\Delta w}^{0} E_{T_2}\,\overline{a_T}\,dl\,\overline{a_T} + \int_{-\Delta h/2}^{0} E_{N_1}\,\overline{a_n}\,dl\,\overline{a_n} + \int_{0}^{\Delta h/2} E_{N_2}\,\overline{a_n}\,dl\,\overline{a_n} = 0$$

$$E_{T_1}\Delta w - E_{N_1}\frac{\Delta h}{2} - E_{N_2}\frac{\Delta h}{2} - E_{T_2}\Delta w$$

$$+ E_{N_1}\frac{\Delta h}{2} + E_{N_2}\frac{\Delta h}{2} = 0$$

Condition I (tangential components)

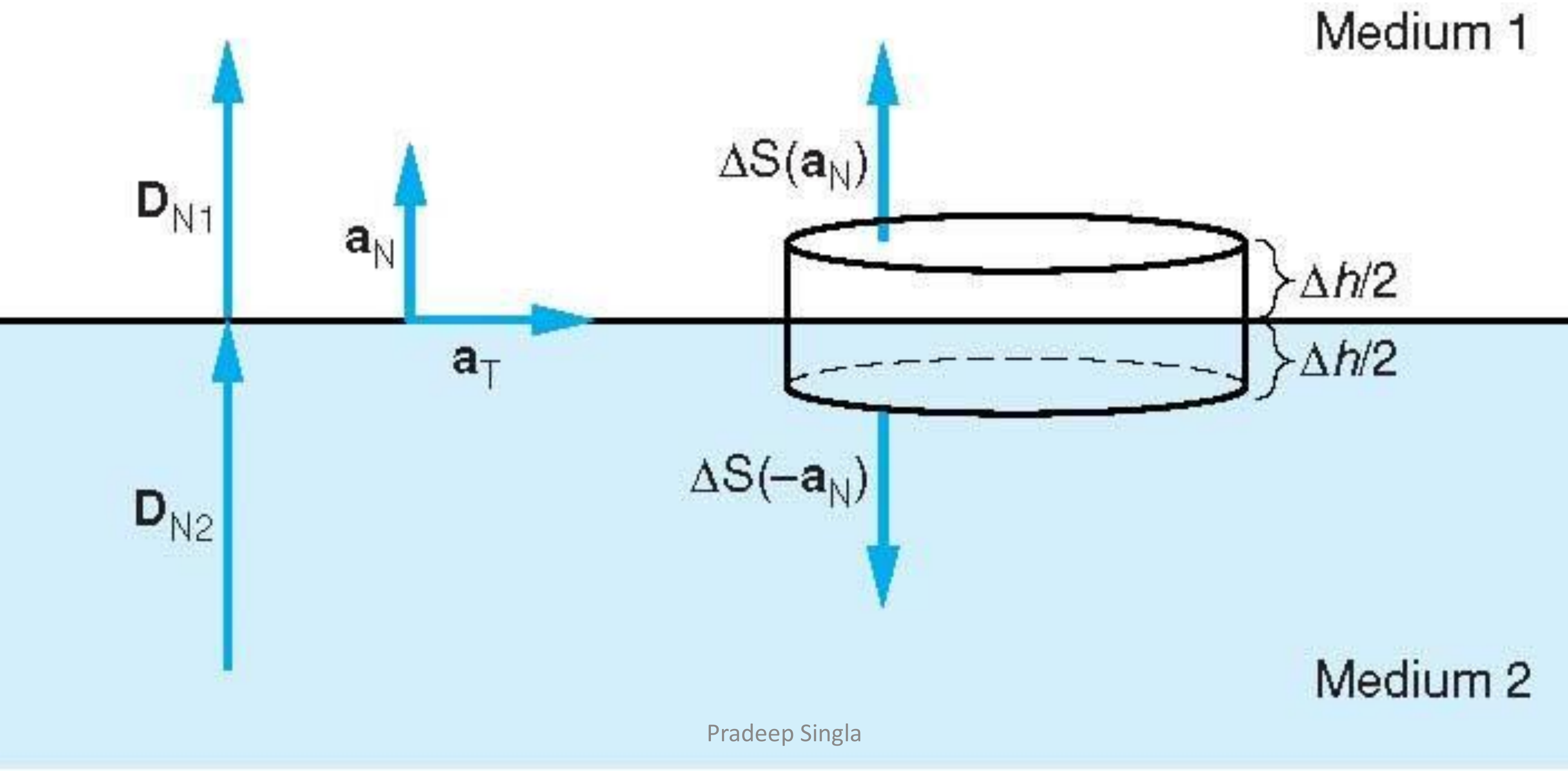
$$E_{T_1} = E_{T_1}$$

$$\frac{D_{T_1}}{\varepsilon_1} = \frac{D_{T_2}}{\varepsilon_2}$$

Perpendicular Components of the Displacement Vector

CONDITION II (normal components)

$$\oint \overline{D} \bullet d\overline{s} = Q$$



$$\oint_s \overline{D} \bullet d\overline{s} = \int_{top} \overline{D} \bullet d\overline{s} + \int_{side} \overline{D} \bullet d\overline{s} + \int_{bottom} \overline{D} \bullet d\overline{s}$$

$$\oint_s \overline{D} \bullet d\overline{s} = \int_{top} D_{N_1} \overline{a_n} ds \overline{a_n} + \int_{bottom} D_{N_1} \overline{a_n} ds (-\overline{a_n}) =$$

$$= D_{N_1} \Delta s - D_{N_2} \Delta s$$

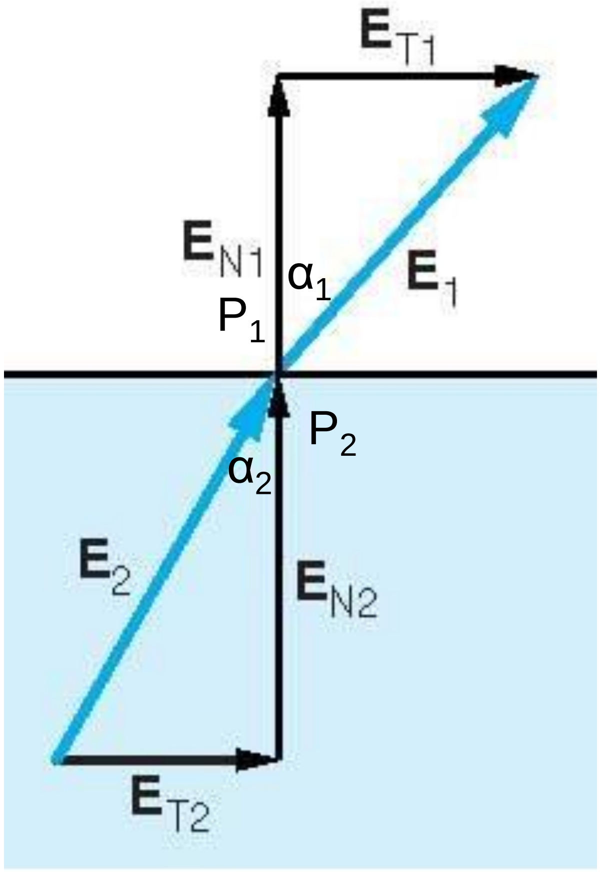
$$Q_{encl} = \int_s \rho_s ds = \rho_s \Delta s$$

Condition II (normal components)

$$a_{21}(D_{N_1} - D_{N_2}) = \rho_s \left(\frac{\mathrm{C}}{\mathrm{m}^2}\right)$$

The normal component of D field is discontinuous across an interface where a surface charge exists, the amount of discontinuity being equal to the surface charge density

Example: Two dielectric media with permittivity ε_1 and respectively ε_2 , are separated by a charge free boundary. The electric field intensity in medium 1 at point P_1 has a magnitude E_1 and makes an angle α_1. Determine the magnitude and direction of the electric field intensity at point P_2 in medium 2.



$$E_{T_1} = E_{T_1}$$

$$D_{N_1} = D_{N_2}$$

$$\text{since } \; a_{12}(D_{N_1} - D_{N_2}) = \rho_s = 0$$

$$E_1 \sin\alpha_1 = E_2 \sin\alpha_2$$

$$\varepsilon_1 E_1 \cos\alpha_1 = \varepsilon_2 E_2 \cos\alpha_2$$

$$\frac{\tan\alpha_1}{\varepsilon_1} = \frac{\tan\alpha_2}{\varepsilon_2}$$

$$\frac{\tan\alpha_1}{\tan\alpha_2} = \frac{\varepsilon_1}{\varepsilon_2}$$

The magnitude of E_2 :

$$\left|\overline{E_2}\right| = \sqrt{E_{T_1}^2 + E_{N_2}^2} = \sqrt{\left(E_2 \sin\alpha_2\right)^2 + \left(E_2 \cos\alpha_2\right)^2} =$$

$$= \sqrt{\left(E_1 \sin\alpha_1\right)^2 + \left(\frac{\varepsilon_1}{\varepsilon_2} E_1 \cos\alpha_1\right)^2}$$

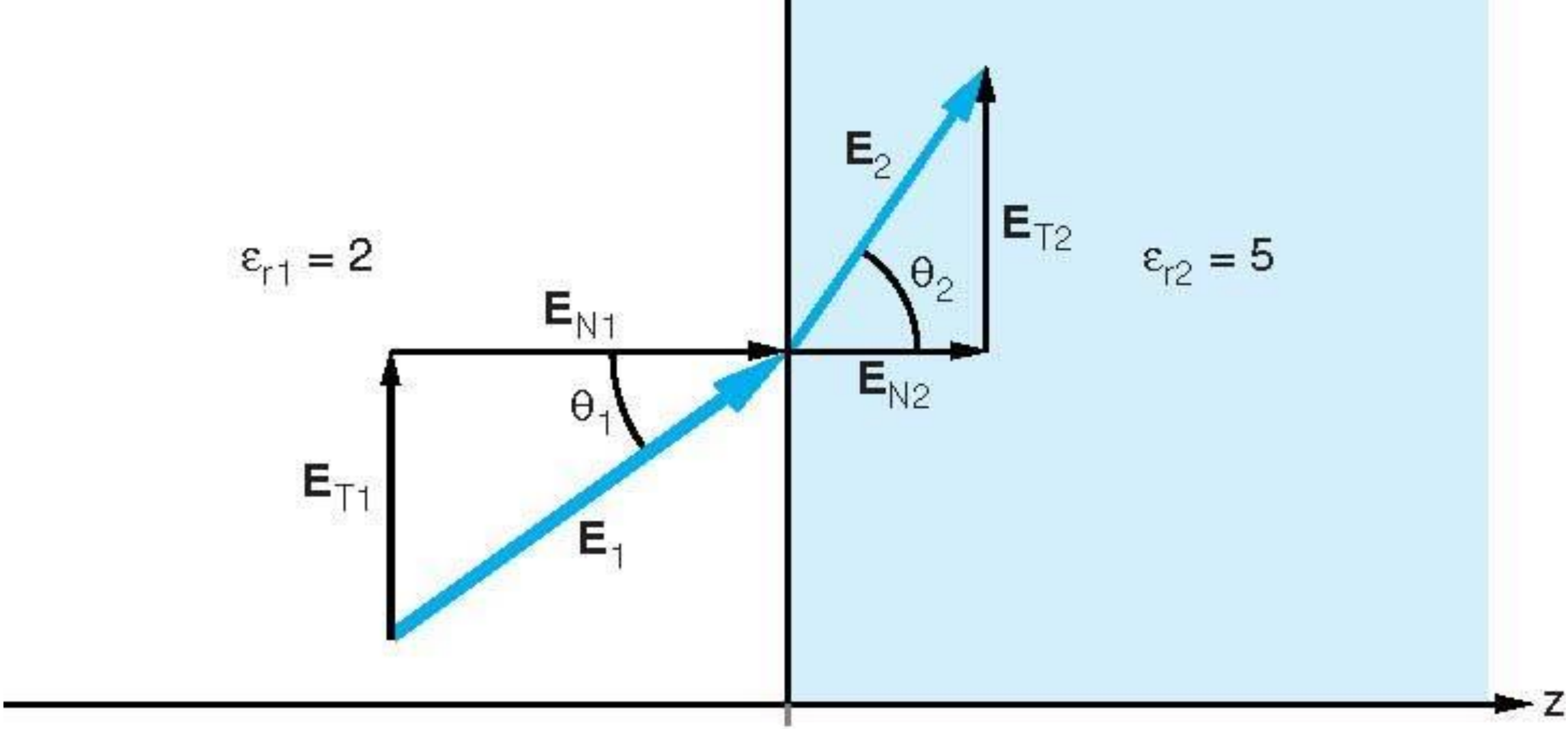
$$E_2 = E_1 \sqrt{\left(\sin\alpha_1\right)^2 + \left(\frac{\varepsilon_1}{\varepsilon_2} E_1 \cos\alpha_1\right)^2}$$

Boundary conditions at a Dielectric/Conductor Interface

-inside a good conductor

$E=0$ → $E_T=0$

$D=0$ → $D_n=\rho_s$



$\mathbf{E}_1 = 3\mathbf{a}_x + 4\mathbf{a}_y + 5\mathbf{a}_z$ (V/m)

Step 1. $\mathbf{E}_{N1} = 5\mathbf{a}_z$

Step 2. $\mathbf{E}_{T1} = 3\mathbf{a}_x + 4\mathbf{a}_y$

Step 3. $\theta_1 = \tan^{-1}\left(\dfrac{|\mathbf{E}_{T1}|}{|\mathbf{E}_{N1}|}\right) = 45°$

Step 5. $\mathbf{D}_{N1} = \varepsilon_{r1}\varepsilon_o\mathbf{E}_{N1} = 10\varepsilon_o\mathbf{a}_z$

Step 9. $\mathbf{E}_2 = 3\mathbf{a}_x + 4\mathbf{a}_y + 2\mathbf{a}_z$ (V/m)

Step 7. $\mathbf{E}_{N2} = \mathbf{D}_{N2}/\varepsilon_{r1}\varepsilon_o = 2\mathbf{a}_z$

Step 4. $\mathbf{E}_{T2} = \mathbf{E}_{T1} = 3\mathbf{a}_x + 4\mathbf{a}_y$

Step 8. $\theta_2 = \tan^{-1}\left(\dfrac{|\mathbf{E}_{T2}|}{|\mathbf{E}_{N2}|}\right) = 68.2°$

Step 6. $\mathbf{D}_{N2} = \mathbf{D}_{N1} = 10\varepsilon_o\mathbf{a}_z$

Practical Problems: Electric Potential

The potential field in a material with $\varepsilon_r = 10.2$ is $V = 12\,xy^2$ (V). Find **E**, **P** and **D**.

$$\mathbf{E} = -\nabla V = -\frac{\partial\left(12xy^2\right)}{\partial x}\mathbf{a}_x - \frac{\partial\left(12xy^2\right)}{\partial y}\mathbf{a}_y = -12y^2\mathbf{a}_x - 24xy\mathbf{a}_y \quad \frac{V}{m}$$

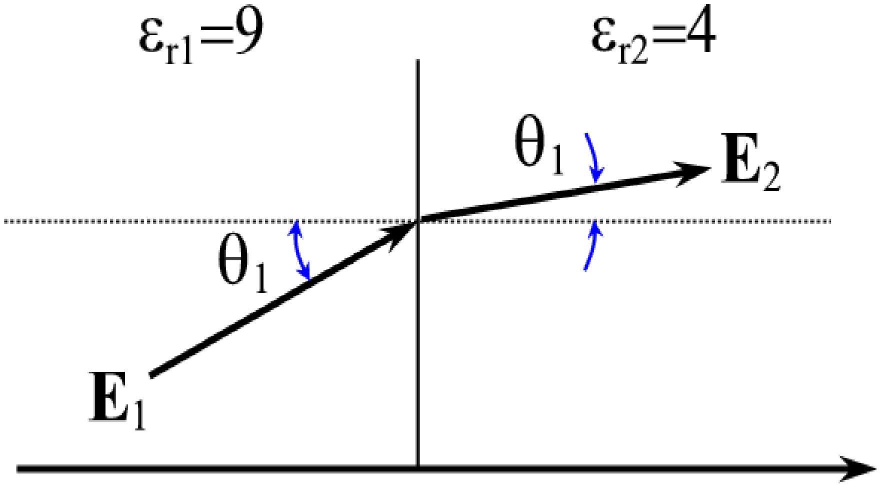
$$\mathbf{D} = \varepsilon_r\varepsilon_o\mathbf{E} = -1.1y^2\mathbf{a}_x - 2.2xy\mathbf{a}_y \quad \frac{nC}{m^2}$$

$$\chi_e = \varepsilon_r - 1 = 9.2$$

$$\mathbf{P} = \chi_e\varepsilon_o\mathbf{E} = (9.2)\left(8.854x10^{-12}\right)\mathbf{E} = -9.8y^2\mathbf{a}_x - 2.00xy\mathbf{a}_y \quad \frac{nC}{m^2}$$

More Practical Problems: Boundary Conditions

For $z \leq 0$, $\varepsilon_{r1} = 9.0$ and for $z > 0$, $\varepsilon_{r2} = 4.0$. If $\mathbf{E}_1$ makes a 30° angle with a normal to the surface, what angle does $\mathbf{E}_2$ make with a normal to the surface?



$$E_{T1} = E_1 \sin\theta_1, \quad E_{T2} = E_2 \sin\theta_2, \text{ and } E_{T1} = E_{T2}$$

also $$\frac{E_{T1}}{D_{N1}} = \frac{E_{T2}}{D_{N2}},$$ Therefore

$$D_{N1} = \varepsilon_{r1}\varepsilon_o E_1 \cos\theta_1, \quad D_{N2} = \varepsilon_{r2}\varepsilon_o E_2 \cos\theta_2,$$

$$\text{and } D_{N1} = D_{N2} \ \left(\text{since } \rho_s = 0\right)$$

and after routine math we find $$\theta_2 = \tan^{-1}\left(\frac{\varepsilon_{r2}}{\varepsilon_{r1}}\tan\theta_1\right)$$

Using this formula we obtain for this problem $\theta_2 = 14°$.